STABLE SOLUTIONS OF SYMMETRIC SYSTEMS ON RIEMANNIAN MANIFOLDS

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ABSTRACT. We examine stable solutions of the following symmetric system on a complete, connected, smooth Riemannian manifold $\mathbb{M}$ without boundary,

$$-\Delta_g u_i = H_i(u_1, \cdots, u_m) \quad \text{on} \quad \mathbb{M},$$

when Δ_g stands for the Laplace-Beltrami operator, $u_i : \mathbb{M} \to \mathbb{R}$ and $H_i \in C^1(\mathbb{R}^m)$ for $1 \le i \le m$. This system is called symmetric if the matrix of partial derivatives of all components of H, that is $\mathbb{H}(u) = (\partial_j H_i(u))_{i,j=1}^m$, is symmetric. We prove a stability inequality and a Poincaré type inequality for stable solutions using the Bochner-Weitzenböck formula. Then, we apply these inequalities to establish Liouville theorems and flatness of level sets for stable solutions of the above symmetric system, under certain assumptions on the manifold and on solutions.

1. INTRODUCTION

Suppose that $\mathbb{M}$ is a complete, connected, smooth, n-dimensional Riemannian manifold without boundary, endowed with a smooth Riemannian metric $g = \{g_{ij}\}$. Consider $u = (u_i)_{i=1}^m$ for $u_i \in C^3(\mathbb{M})$ that satisfies

$$(1.1) \qquad -\Delta_g u_i = H_i(u) \quad \text{on} \quad \mathbb{M},$$

where Δ_g stands for the Laplace-Beltrami operator and $H_i \in C^1(\mathbb{R}^m)$ for $i = 1, \ldots, m$. Let $f : \mathbb{M} \to \mathbb{R}$ be a function in $C^3(\mathbb{M})$. Then, the Riemannian gradient and the Laplace-Beltrami operator are given by

$$(1.2) \qquad (\nabla_g f)_i = g^{ij} \partial_j f,$$

and

$$(1.3) \qquad \Delta_g f = \text{div}_g(\nabla_g f) = \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} g^{ij} \partial_j f \right).$$

In this article, we frequently refer to the Bochner-Weitzenböck formula that is

$$(1.4) \qquad \frac{1}{2} \Delta_g |\nabla_g f|^2 = |\mathcal{H}_f|^2 + \nabla_g \Delta_g f \cdot \nabla_g f + \text{Ric}_g(\nabla_g f, \nabla_g f).$$

Here $\mathcal{H}_f$ stands for the Hessian of f as the symmetric 2-tensor given in a local patch that is

$$(1.5) \qquad (\mathcal{H}_f)_{ij} := \partial_{ij} f - \Gamma_{ij}^k \partial_k f,$$

where Γ_{ij}^k is the Christoffel symbol that is

$$(1.6) \qquad \Gamma_{ij}^k = \frac{1}{2} g^{hk} (\partial_i g_{hj} + \partial_j g_{ih} - \partial_h g_{ij}).$$

Note that from the definition of the Hessian one can see that

$$(1.7) \qquad |\nabla_g |\nabla_g f||^2 \le |\mathcal{H}_f|^2.$$

The equality holds at $p \in \mathbb{M} \cap \{\nabla_g f \ne 0\}$ if and only if there exists $\kappa_k : \mathbb{M} \to \mathbb{R}$ for each $k = 1, \cdots, n$ such that

$$(1.8) \qquad \nabla_g (\nabla_g f)_k(p) = \kappa_k(p) \nabla_g f(p),$$

see [15, 27, 29] for details. Here, we provide the definition of parabolic manifolds. We refer interested readers to [8, 25, 30, 34] for more information.

Definition 1.1. *A manifold $\mathbb{M}$ is called parabolic when for every point $p \in \mathbb{M}$ there exists a precompact neighborhood M_p of p in $\mathbb{M}$ such that for an arbitrary positive ϵ there exists a function $f_\epsilon \in C_c^\infty(\mathbb{M})$ that $f_\epsilon(q) = 1$ for all $q \in M_p$ and $\int_{\mathbb{M}} |\nabla_g f_\epsilon|^2 dV_g \le \epsilon$.*

For the sake of simplicity, we use the notation $\partial_j H_i(u) = \frac{\partial H_i(u)}{\partial u_j}$ and we assume that

$$(1.9) \qquad \partial_i H_j(u) \partial_j H_i(u) > 0,$$

for $1 \le i, j \le m$. In addition, B_R stands for a geodesic ball of radius $R > 0$ centred at a given point of $\mathbb{M}$ and $|B_R|$ is its volume with respect to the volume element dV_g. We now provide the notion of stable solutions.

Definition 1.2. *A solution $u = (u_i)_i$ of (1.1) is said to be stable when there exist a sequence of functions $\zeta = (\zeta_i)_i$ where each $\zeta_i \in C^3(\mathbb{M})$ does not change sign and a nonnegative constant λ such that the following linearized system holds*

$$(1.10) \qquad -\Delta \zeta_i = \sum_{j=1}^n \partial_j H_i(u) \zeta_j + \lambda \zeta_i \quad on \ \ \mathbb{M},$$

for all $i = 1, \cdots, m$. In addition, we assume that $\partial_j H_i(u) \zeta_i \zeta_j > 0$ for all $1 \le i, j \le m$.

For the case of $\mathbb{M} = \mathbb{R}^n$ and scalar equations, that is when $m = 1$, the above notion of stability was derived in connections with the De Giorgi's conjecture and it is used in the literature extensively, see [2, 3, 11, 14–17, 24, 33, 35]. The latter conjecture [10], given in 1978, states that bounded monotone solutions of the Allen-Cahn equation must be hyperplane, see [2, 3, 15, 24, 33] and references therein. For the case of system of equations, that is when $m \ge 2$

$$(1.11) \qquad -\Delta u_i = H_i(u_1, \cdots, u_m) \ \text{on} \ \mathbb{R}^n,$$

the notion of stability is given in [20–22] and references therein. We also refer interested readers to [4, 5, 18, 38] for the following two-component elliptic system, originated in the Bose-Einstein condensation and nonlinear optics,

$$(1.12) \qquad \begin{cases} -\Delta u = H_1(u, v) & \text{on} \ \mathbb{R}^n, \\ -\Delta v = H_2(u, v) & \text{on} \ \mathbb{R}^n, \end{cases}$$

when $H_1(u, v) = -uv^2$ and $H_2(u, v) = -vu^2$. For this system, nonnegative monotone solutions, which are $u_{x_n}(x) v_{x_n}(x) < 0$ in $\mathbb{R}^n$, are of interests. Straightforward

computations show that monotone solutions satisfy (1.10) for $\zeta_1 = u_{x_n}$, $\zeta_2 = v_{x_n}$ and $\lambda = 0$. Note also that $\partial_v H_1 = \partial_u H_2 < 0$ and $\partial_v H_1 \partial_u H_2 > 0$. For a similar notion of stability, we refer interested readers to [1, 22] for the Allen-Cahn system and to [9, 12, 13, 20, 21, 31] for systems with general nonlinearities on bounded and unbounded domains.

We now provide the notion of symmetric systems introduced in [21] when $\mathbb{M} = \mathbb{R}^n$. Symmetric systems play a fundamental role throughout this paper when we study system (1.1) with a general nonlinearity $H(u) = (H_i(u))_{i=1}^m$.

Definition 1.3. *We call system (1.1) symmetric if the matrix of gradient of all components of H that is*

$$(1.13) \qquad \mathbb{H} := (\partial_i H_j(u))_{i,j=1}^m,$$

is symmetric.

Here is how this article is structured. In Section 2, we provide a stability inequality for stable solutions of (1.1). Applying this stability inequality we prove a weighted Poincaré type inequality for stable solutions of system (1.1). In Section 3, we provide applications of inequalities provided in Section 2. To be mathematically more precise, we establish various Liouville theorems regarding stable solutions as well as a rigidity result concerning level sets of stable solutions.

2. Inequalities for stable solutions

We start this section by an inequality for stable solutions of (1.1). For the case of a scalar equations, this inequality is used in the literature extensively to study symmetry properties, regularity theory, Liouville theorems, etc. regarding stable solutions. We refer interested readers to [2, 3, 7, 15–17, 23, 24, 32] for more information. For the case of system of equations and when $\mathbb{M} = \mathbb{R}^n$, this inequality is given in [9, 21, 22].

Lemma 2.1. *Let u denote a stable solution of (1.1). Then*

$$(2.1) \qquad \sum_{i,j=1}^m \int_{\mathbb{M}} \sqrt{\partial_j H_i(u) \partial_i H_j(u)}\, \phi_i \phi_j\, dV_g \leq \sum_{i=1}^m \int_{\mathbb{M}} |\nabla_g \phi_i|^2 dV_g,$$

for any $\phi = (\phi_i)_{i=1}^m$ where $\phi_i \in C_c^1(\mathbb{M})$ for $1 \leq i \leq m$.

Proof. The fact that u is a stable solution implies that there exist a sequence $\zeta = (\zeta_i)_{i=1}^m$ and a nonnegative constant λ such that for all $i = 1, \cdots, m$

$$(2.2) \qquad -\Delta_g \zeta_i = \sum_{j=1}^n \partial_j H_i(u) \zeta_j + \lambda \zeta_i \ \text{ on } \ \mathbb{M}.$$

Consider a sequence of test functions $\phi = (\phi_i)_i^m$ where $\phi_i \in C_c^1(\mathbb{M})$ for $1 \leq i \leq m$. Multiplying both sides of (2.2) with $\frac{\phi_i^2}{\zeta_i}$ and integrating, we get

$$(2.3) \qquad \lambda \int_{\mathbb{M}} \phi_i^2 + \sum_{j=1}^n \int_{\mathbb{M}} \partial_j H_i(u) \zeta_j \frac{\phi_i^2}{\zeta_i} dV_g = \int_{\mathbb{M}} -\frac{\Delta_g \zeta_i}{\zeta_i} \phi_i^2 dV_g.$$

Applying the fact that

$$(2.4) \qquad 2\frac{\phi_i}{\zeta_i} \nabla_g \zeta_i \cdot \nabla_g \phi_i - |\nabla_g \zeta_i|^2 \frac{\phi_i^2}{\zeta_i^2} \leq |\nabla_g \phi_i|^2,$$

we obtain

$$(2.5) \qquad \int_{\mathbb{M}} -\frac{\Delta_g \zeta_i}{\zeta_i} \phi_i^2 dV_g \leq \int_{\mathbb{M}} |\nabla_g \phi_i|^2 dV_g,$$

for each $i = 1, \cdots, m$. Note also that here we have applied the divergence theorem for the Laplace-Beltrami operator. Combining (2.5) and (2.3), we end up with

$$(2.6) \qquad \sum_{i,j=1}^{n} \int_{\mathbb{M}} \partial_j H_i(u) \zeta_j \frac{\phi_i^2}{\zeta_i} dV_g \leq \sum_{i=1}^{n} \int_{\mathbb{M}} |\nabla_g \phi_i|^2 dV_g.$$

For the left-hand side of (2.6), straightforward calculations show that

$$
\begin{aligned}
\sum_{i,j=1}^{m} \int_{\mathbb{M}} \partial_j H_i(u) \zeta_j \frac{\phi_i^2}{\zeta_i} dV_g \;=\;& \sum_{i<j} \int_{\mathbb{M}} \partial_j H_i(u) \zeta_j \frac{\phi_i^2}{\zeta_i} dV_g + \sum_{i>j} \int_{\mathbb{M}} \partial_j H_i(u) \zeta_j \frac{\phi_i^2}{\zeta_i} dV_g \\
&+ \sum_{i=1}^{m} \int_{\mathbb{M}} \partial_i H_i(u) \phi_i^2 dV_g \\
=\;& \sum_{i<j} \int_{\mathbb{M}} \partial_j H_i(u) \zeta_j \frac{\phi_i^2}{\zeta_i} dV_g + \sum_{i<j} \int_{\mathbb{M}} \partial_i H_j(u) \zeta_i \frac{\phi_j^2}{\zeta_j} dV_g \\
&+ \sum_{i=1}^{m} \int_{\mathbb{M}} \partial_i H_i(u) \phi_i^2 dV_g \\
=\;& \sum_{i<j} \int_{\mathbb{M}} \left(\partial_j H_i(u) \zeta_j \frac{\phi_i^2}{\zeta_i} + \partial_i H_j(u) \zeta_i \frac{\phi_j^2}{\zeta_j} \right) dV_g \\
&+ \sum_{i=1}^{m} \int_{\mathbb{M}} \partial_i H_i(u) \phi_i^2 dV_g \\
\geq\;& 2 \sum_{i<j} \int_{\mathbb{M}} \sqrt{\partial_j H_i(u) \partial_i H_j(u)} \phi_i \phi_j dV_g + \sum_{i=1}^{m} \int_{\mathbb{M}} \partial_i H_i(u) \phi_i^2 dV_g \\
(2.7) \qquad =\;& \sum_{i,j=1}^{m} \int_{\mathbb{M}} \sqrt{\partial_j H_i(u) \partial_i H_j(u)} \phi_i \phi_j dV_g.
\end{aligned}
$$

This finishes the proof.

$\square$

Applying the stability inequality (2.1), given in Lemma 2.1, we prove a weighted Poincaré inequality as the following theorem. Note that for the case of scalar equations a similar inequality is established in [36] and used in [7, 15, 37], in the Euclidean sense, and in [16, 17] on Riemannian manifolds. For the case of system of equations, that is when $m \geq 2$, this inequality was derived in [22] in the Euclidean sense and later used in [12, 13, 20] and references therein.

Theorem 2.1. *Assume that $u \in C^3(\mathbb{M})$ is a stable solution of (1.1) where $m, n \geq 1$. Then, the following inequality holds for any $\eta = (\eta_k)_{k=1}^m \in C_c^1(\mathbb{M})$*

$$(2.8) \quad \sum_{i=1}^m \int_{\mathbb{M}} \left(Ric_g(\nabla_g u_i, \nabla_g u_i) + |\mathcal{H}_{u_i}|^2 - |\nabla_g|\nabla_g u_i||^2 \right) \eta_i^2 dV_g$$

$$+ \sum_{i \neq j}^m \int_{\mathbb{M}} \left(\sqrt{\partial_j H_i(u) \partial_i H_j(u)} |\nabla_g u_i||\nabla_g u_j|\eta_i \eta_j - \partial_j H_i(u) \nabla_g u_i \cdot \nabla_g u_j \eta_i^2 \right) dV_g$$

$$\leq \sum_{i=1}^m \int_{\mathbb{M}} |\nabla_g u_i|^2 |\nabla_g \eta_i|^2 dV_g.$$

Proof. Test the stability inequality (2.1) on $\phi_i = |\nabla_g u_i|\eta_i$ for the sequence of test functions $\eta = (\eta_i)_{i=1}^m$ and $\eta_i \in C_c^1(\mathbb{M})$, to get

$$(2.9) \quad I \; := \; \sum_{i,j=1}^m \int_{\mathbb{M}} \sqrt{\partial_j H_i(u) \partial_i H_j(u)} |\nabla_g u_i||\nabla_g u_j|\eta_i \eta_j dV_g$$

$$\leq \; \sum_{i=1}^m \int_{\mathbb{M}} |\nabla_g(|\nabla_g u_i|\eta_i)|^2 dV_g =: J.$$

We rewrite I as

$$(2.10) \quad I \; = \; \sum_{i=1}^m \int_{\mathbb{M}} |\partial_i H_i(u)||\nabla_g u_i|^2 \eta_i^2 dV_g$$

$$+ \sum_{i \neq j}^m \int_{\mathbb{M}} \sqrt{\partial_j H_i(u) \partial_i H_j(u)} |\nabla_g u_i||\nabla_g u_j|\eta_i \eta_j dV_g.$$

A simple integration by parts implies

$$(2.11) \quad J = \sum_{i=1}^m \int_{\mathbb{M}} |\nabla_g|\nabla_g u_i||^2 \eta_i^2 + |\nabla_g u_i|^2 |\nabla_g \eta_i|^2 + 2(\eta_i|\nabla_g u_i|)\nabla_g|\nabla_g u_i| \cdot \nabla_g \eta_i dV_g$$

$$= \sum_{i=1}^m \int_{\mathbb{M}} |\nabla_g|\nabla_g u_i||^2 \eta_i^2 + |\nabla_g u_i|^2 |\nabla_g \eta_i|^2 - \frac{1}{2}\eta_i^2 \Delta_g|\nabla_g u_i|^2 dV_g.$$

We now apply the Bochner-Weitzenböck formula, given in (1.4), to have

$$(2.12) \quad J = \sum_{i=1}^m \int_{\mathbb{M}} |\nabla_g|\nabla_g u_i||^2 \eta_i^2 + |\nabla_g u_i|^2 |\nabla_g \eta_i|^2 dV_g$$

$$- \sum_{i=1}^m \int_{\mathbb{M}} |\mathcal{H}_{u_i}|^2 \eta_i^2 + \nabla_g \Delta_g u_i \cdot \nabla_g u_i \eta_i^2 + Ric_g(\nabla_g u_i, \nabla_g u_i)\eta_i^2 dV_g.$$

Combining (2.12), (2.10) and (2.9) implies

$$(2.13) \qquad \sum_{i=1}^{m} \int_{\mathbb{M}} |\partial_i H_i(u)||\nabla_g u_i|^2 \eta_i^2 dV_g$$

$$\leq \; -\sum_{i \neq j}^{m} \int_{\mathbb{M}} \sqrt{\partial_j H_i(u) \partial_i H_j(u)} |\nabla_g u_i||\nabla_g u_j|\eta_i \eta_j dV_g$$

$$+\sum_{i=1}^{m} \int_{\mathbb{M}} |\nabla_g|\nabla_g u_i||^2 \eta_i^2 + |\nabla_g u_i|^2 |\nabla_g \eta_i|^2 dV_g$$

$$-\sum_{i=1}^{m} \int_{\mathbb{M}} |\mathcal{H}_{u_i}|^2 \eta_i^2 + \nabla_g \Delta_g u_i \cdot \nabla_g u_i \eta_i^2 + \mathrm{Ric}_g(\nabla_g u_i, \nabla_g u_i)\eta_i^2 dV_g.$$

On the other hand, for each $i = 1, 2, ..., m$ differentiating both sides of the ith equation in (1.1) and multiplying with $\nabla_g u_i \eta_i^2$ we get

$$(2.14) \qquad -\nabla_g u_i \cdot \nabla_g \Delta_g u_i \eta_i^2 = \sum_{j=1}^{m} \partial_j H_i(u) \nabla_g u_j \cdot \nabla_g u_i \eta_i^2.$$

Integrating both sides of the above equation gives

$$(2.15) \qquad I_1 \; := \; -\sum_{i=1}^{m} \int_{\mathbb{M}} \nabla_g u_i \cdot \nabla_g \Delta_g u_i \eta_i^2 dV_g$$

$$= \; \sum_{i,j=1}^{m} \int_{\mathbb{M}} \partial_j H_i(u) \nabla_g u_j \cdot \nabla_g u_i \eta_i^2 dV_g =: J_1.$$

Note that J_1 can be rewritten as

$$(2.16) \qquad J_1 = \sum_{i=1}^{m} \int_{\mathbb{M}} \partial_i H_i(u)|\nabla_g u_i|^2 \eta_i^2 dV_g + \sum_{i \neq j}^{m} \int_{\mathbb{M}} \partial_j H_i(u)\nabla_g u_j \cdot \nabla_g u_i \eta_i^2 dV_g.$$

From (2.16) and (2.15) we get

$$(2.17) \; \sum_{i=1}^{m} \int_{\mathbb{M}} \partial_i H_i(u)|\nabla_g u_i|^2 \eta_i^2 dV_g \; = \; -\sum_{i \neq j}^{m} \int_{\mathbb{M}} \partial_j H_i(u)\nabla_g u_j \cdot \nabla_g u_i \eta_i^2 dV_g$$

$$-\sum_{i=1}^{m} \int_{\mathbb{M}} \nabla_g u_i \cdot \nabla_g \Delta_g u_i \eta_i^2 dV_g.$$

Combining (2.13) and (2.17), completes the proof.

$\square$

3. Liouville theorems

In this section, we provide various Liouville theorems for solutions of (1.1) as consequences of the Poincaré inequality given in Theorem 2.1. We start with the following Liouville theorem.

Theorem 3.1. *Suppose that $u = (u_i)_{i=1}^{m}$ is a stable solution of symmetric system (1.1) where the Ricci curvature is nonnegative and Ric_g is not identically zero. Assume also that one of the following conditions holds*

    (i) $\mathbb{M}$ *is compact.*

(ii) $\mathbb{M}$ *is complete and parabolic and* $|\nabla_g u_i| \in L^\infty(\mathbb{M})$ *for each* $i = 1, \cdots, m$.

Then, each u_i *must be constant for* $i = 1, \cdots, m$.

Proof. We start with Part (i). If $\mathbb{M}$ is compact, then we set $\eta_i = 1$ in the stability inequality to get

$$(3.1) \qquad \sum_{i=1}^m \int_{\mathbb{M}} \left(Ric_g(\nabla_g u_i, \nabla_g u_i) + |\mathcal{H}_{u_i}|^2 - |\nabla_g|\nabla_g u_i||^2 \right) dV_g$$

$$+ \sum_{i \neq j} \int_{\mathbb{M}} (|\partial_j H_i(u)||\nabla_g u_i||\nabla_g u_j| - \partial_j H_i(u) \nabla_g u_i \cdot \nabla_g u_j) \, dV_g \leq 0.$$

Note that

$$(3.2) \qquad |\partial_j H_i(u)||\nabla_g u_i||\nabla_g u_j| \geq \partial_j H_i(u) \nabla_g u_i \cdot \nabla_g u_j.$$

This and (3.1) imply that

$$(3.3) \qquad \sum_{i=1}^m \int_{\mathbb{M}} \left[Ric_g(\nabla_g u_i, \nabla_g u_i) + |\mathcal{H}_{u_i}|^2 - |\nabla_g|\nabla_g u_i||^2 \right] dV_g \leq 0.$$

Note that the Ricci curvature is nonnegative and the following inequality holds

$$(3.4) \qquad |\nabla_g|\nabla_g u_i||^2 \leq |\mathcal{H}_{u_i}|^2.$$

From this and (3.3), we get

$$(3.5) \qquad \int_{\mathbb{M}} Ric_g(\nabla_g u_i, \nabla_g u_i) dV_g = \int_{\mathbb{M}} \left[|\mathcal{H}_{u_i}|^2 - |\nabla_g|\nabla_g u_i||^2 \right] dV_g = 0.$$

This implies that for every $p \in \mathbb{M} \cap \{\nabla_g u_i \neq 0\}$, we have

$$(3.6) \qquad |\nabla_g|\nabla_g u_i||^2(p) = |\mathcal{H}_{u_i}|^2(p),$$

and

$$(3.7) \qquad Ric_g(\nabla_g u_i, \nabla_g u_i)(p) = 0.$$

Note that (3.6) implies that

$$(3.8) \qquad \nabla_g(\nabla_g u_i)_k(p) = \kappa_k(p) \nabla_g u_i(p).$$

From the assumptions on Ric_g, we conclude that Ric_g is positive definite in some open subset of $\mathbb{M}$. This and (3.7) imply that for each i, we have $\nabla_g u_i(p) = 0$ for any p in an open subset of $\mathbb{M}$. Therefore, the unique continuation principle [28] implies that each u_i must be constant on $\mathbb{M}$.

We now consider Part (ii). Since $\mathbb{M}$ is parabolic, from Definition 1.1 there exist M_p and f_ϵ such that $\int_{\mathbb{M}} |\nabla_g f_\epsilon|^2 dV_g \leq \epsilon$. Set $\eta_i := f_\epsilon$ in the Poincaré inequality

(2.8) to get

$$(3.9) \qquad \sum_{i=1}^{m} \int_{M_p} \left(Ric_g(\nabla_g u_i, \nabla_g u_i) + |\mathcal{H}_{u_i}|^2 - |\nabla_g|\nabla_g u_i||^2 \right) dV_g$$

$$+ \sum_{i \neq j} \int_{M_p} \left(|\partial_j H_i(u)||\nabla_g u_i||\nabla_g u_j| - \partial_j H_i(u)\nabla_g u_i \cdot \nabla_g u_j \right) dV_g$$

$$\leq \quad \sum_{i=1}^{m} \int_{\mathbb{M}} \left(Ric_g(\nabla_g u_i, \nabla_g u_i) + |\mathcal{H}_{u_i}|^2 - |\nabla_g|\nabla_g u_i||^2 \right) f_\epsilon dV_g$$

$$+ \sum_{i \neq j} \int_{\mathbb{M}} \left(|\partial_j H_i(u)||\nabla_g u_i||\nabla_g u_j| - \partial_j H_i(u)\nabla_g u_i \cdot \nabla_g u_j \right) f_\epsilon^2 dV_g$$

$$\leq \quad \sum_{i=1}^{m} \int_{\mathbb{M}} |\nabla_g u_i|^2 |\nabla_g f_\epsilon|^2 dV_g$$

$$\leq \quad \sum_{i=1}^{m} ||\nabla_g u_i||_{L^\infty(\mathbb{M})}^2 \int_{\mathbb{M}} |\nabla_g f_\epsilon|^2 dV_g$$

$$\leq \quad \epsilon \max_{i=1}^{m} \{ ||\nabla_g u_i||_{L^\infty(\mathbb{M})}^2 \}.$$

Sending $\epsilon \to 0$ and using the fact that the inequality (3.2) holds, we get the following estimate that is a counterpart of (3.3) on M_p,

$$(3.10) \qquad \sum_{i=1}^{m} \int_{M_p} \left[Ric_g(\nabla_g u_i, \nabla_g u_i) + |\mathcal{H}_{u_i}|^2 - |\nabla_g|\nabla_g u_i||^2 \right] dV_g \leq 0.$$

Applying similar arguments given for Part (i), completes the proof.

$\square$

The next theorem deals with compact manifolds when Ric_g is identically zero. We show that when the manifold is compact and Ric_g is precisely zero, then stable solutions must be constant. Comparing Theorem 3.1 with the following theorem may help us better understand the role of Ric_g. We refer interested readers to [26] for more information.

Theorem 3.2. *Let* $u = (u_i)_{i=1}^{m}$ *be a stable solution for symmetric system (1.1). Suppose that* $\mathbb{M}$ *is a compact and connected Riemannian manifold and* Ric_g *is zero. Then, each* u_i *is constant for* $i = 1, \cdots, m$.

Proof. For each i, differentiate (1.1) and multiply with $\nabla_g u_i$ to get

$$(3.11) \qquad \nabla_g u_i \cdot \nabla_g \Delta_g u_i = - \sum_{j=1}^{m} \partial_j H_i(u) \nabla_g u_j \cdot \nabla_g u_i.$$

Note that (1.7) implies that for each i

$$(3.12) \qquad |\nabla_g|\nabla_g u_i||^2 \leq |\mathcal{H}_{u_i}|^2.$$

Adding (3.11) and (3.12), we get

$$(3.13) \quad |\mathcal{H}_{u_i}|^2 + \nabla_g u_i \cdot \nabla_g \Delta_g u_i \geq |\nabla_g|\nabla_g u_i||^2 - \sum_{j=1}^{m} \partial_j H_i(u) \nabla_g u_j \cdot \nabla_g u_i.$$

Since Ric_g is assumed to be zero, the Bochner-Weitzenböck formula (1.4) implies

$$(3.14) \qquad \frac{1}{2}\Delta_g|\nabla_g u_i|^2 = |\mathcal{H}_{u_i}|^2 + \nabla_g \Delta_g u_i \cdot \nabla_g u_i.$$

Combining this and (3.13) we get

$$(3.15) \qquad \frac{1}{2}\Delta_g|\nabla_g u_i|^2 \geq |\nabla_g|\nabla_g u_i||^2 - \sum_{j=1}^m \partial_j H_i(u)\nabla_g u_j \cdot \nabla_g u_i.$$

Integrating this on $\mathbb{M}$ and considering the fact that $\mathbb{M}$ does not have a boundary, for each i, we obtain

$$(3.16) \qquad \int_{\mathbb{M}} \left[|\nabla_g|\nabla_g u_i||^2 - \sum_{j=1}^m \partial_j H_i(u)\nabla_g u_j \cdot \nabla_g u_i \right] dV_g \leq 0.$$

Taking summation on index $1 \leq i \leq m$ for (3.16), yields

$$(3.17) \qquad \begin{aligned} I_1 &:= \sum_{i=1}^m \int_{\mathbb{M}} \left[|\nabla_g|\nabla_g u_i||^2 dV_g - \partial_i H_i(u)|\nabla_g u_i|^2 \right] dV_g \\ &\leq \sum_{i \neq j}^m \int_{\mathbb{M}} \partial_j H_i(u)\nabla_g u_j \cdot \nabla_g u_i dV_g =: I_2. \end{aligned}$$

On the other hand, from the stability inequality (2.1) for symmetric systems, we have

$$(3.18) \qquad \begin{aligned} I_3(\phi) &:= \sum_{i=1}^m \int_{\mathbb{M}} \left[|\nabla_g\phi_i|^2 - \partial_i H_i(u)\phi_i^2 \right] dV_g \\ &\geq \sum_{i \neq j}^m \int_{\mathbb{M}} |\partial_j H_i(u)|\phi_i\phi_j dV_g =: I_4(\phi), \end{aligned}$$

when $\phi = (\phi_i)_{i=1}^m$ is a sequence of test functions. Setting $\phi_i = |\nabla_g u_i|$ in the above $I_3(\phi)$ and $I_4(\phi)$, we get

$$(3.19) \qquad I_1 = I_3(\phi) \quad \text{and} \quad I_2 \leq I_4(\phi).$$

This implies that $\phi = (\phi_i)_{i=1}^m$ when $\phi_i = |\nabla_g u_i|$ is the minimizer of the energy $J(\phi) := I_3(\phi) - I_4(\phi) \geq 0$. Therefore,

$$(3.20) \qquad -\Delta_g|\nabla_g u_i| = \sum_{j=1}^n \partial_j H_i(u)|\nabla_g u_j| \quad \text{on} \quad \mathbb{M},$$

for each $i = 1, \cdots, m$. From the compactness of $\mathbb{M}$, there exists a point $\bar{p}_i \in \mathbb{M}$ such that $u_i(\bar{p}_i) = \max_{\mathbb{M}} u_i$. This implies that $\nabla_g u_i(\bar{p}_i) = 0$. From the strong maximum principle, we get $|\nabla_g u_i| \equiv 0$ for each $i = 1, \cdots, m$. This completes the proof. $\qquad\square$

We now provide the following Liouville theorem in lower dimensions for any nonlinearity $H = (H_i)_{i=1}^m$ with nonnegative components H_i. Note that for a complete, connected, Riemannian manifold $\mathbb{M}$ with nonnegative Ricci curvature and dimension n, the volume of a ball of radius R, denoted $|B_R|$, is bounded by R^n. This fact implies that the following Liouville theorem is valid for $n < 4$.

Theorem 3.3. *Suppose that $u = (u_i)_{i=1}^m$ is a bounded stable solution of (1.1) where $H_i \geq 0$ for each $i = 1, \cdots, m$. Let the following decay estimate hold*

$$(3.21) \qquad\qquad \liminf_{R \to \infty} R^{-4}|B_R| = 0.$$

Assume also that the Ricci curvature of $\mathbb{M}$ is nonnegative and Ric_g is not identically zero. Then, each u_i must be constant for $i = 1, \cdots, m$.

Proof. Suppose that $\zeta \in C_c^\infty([-2,2]) \to [0,1]$ where $\zeta \equiv 1$ on $[-1,1]$. For $p \in \mathbb{M}$ and $R > 0$, set

$$(3.22) \qquad\qquad \zeta_R(p) = \zeta\left(\frac{d_g(p)}{R}\right),$$

where d_g is the geodesic distance. Therefore, $\zeta_R \in C_c^\infty(\mathbb{M})$ satisfies $\zeta_R = 1$ on B_R and $\zeta_R = 0$ on $\mathbb{M} \setminus B_{2R}$ and

$$(3.23) \qquad\qquad ||\nabla_g \zeta_R||_{L^\infty(B_{2R} \setminus B_R)} \leq \frac{C}{R}.$$

From the assumptions, each u_i is bounded. Multiplying both sides of system (1.1) with $(u_i - ||u_i||_{L^\infty(\mathbb{M})})\zeta_R^2$ and from the fact that $H_i \geq 0$, we get

$$(3.24) \qquad\qquad H_i(u)[u_i - ||u_i||_{L^\infty(\mathbb{M})}]\zeta_R^2 \leq 0 \ \text{ in } \ \mathbb{M}.$$

From this and the equation (1.1), we have

$$(3.25) \qquad\qquad -\Delta_g u_i(u_i - ||u_i||_{L^\infty(\mathbb{M})})\zeta_R^2 \leq 0 \ \text{ in } \ \mathbb{M}.$$

Doing integration by parts, for each $i = 1 \cdots, m$, we obtain

$$(3.26) \qquad \int_{B_{2R}} |\nabla_g u_i|^2 \zeta_R^2 dV_g \leq 2 \int_{B_{2R}} |\nabla_g u_i||\nabla \zeta_R|(||u_i||_{L^\infty(\mathbb{M})} - u_i)\zeta_R dV_g.$$

Applying the Cauchy-Schwarz inequality yields

$$(3.27) \qquad\qquad \int_{B_{2R}} |\nabla_g u_i|^2 \zeta_R^2 dV_g \leq C \int_{B_{2R}} |\nabla_g \zeta_R|^2 dV_g,$$

where the constant C is independent from R. From the definition of ζ_R and applying (3.27), we get

$$(3.28) \qquad \begin{aligned} \sum_{i=1}^m \int_{B_R} |\nabla_g u_i|^2 dV_g & \leq \sum_{i=1}^m \int_{B_{2R}} |\nabla_g u_i|^2 \zeta_R^2 dV_g \\ & \leq C \int_{B_{2R}} |\nabla_g \zeta_R|^2 dV_g \leq C\frac{|B_{2R}|}{R^2}. \end{aligned}$$

We now apply the Poincaré inequality (2.8) with the test function $\eta_i = \zeta_R$. In the light of (3.28), the right-hand side of (2.8) becomes

$$(3.29) \ \begin{aligned} \liminf_{R \to \infty} \sum_{i=1}^m \int_{\mathbb{M}} |\nabla_g u_i|^2 |\nabla_g \eta_i|^2 dV_g & \leq \liminf_{R \to \infty} \frac{C}{R^2} \sum_{i=1}^m \int_{B_{2R}} |\nabla_g u_i|^2 dV_g \\ & \leq C \liminf_{R \to \infty} \frac{|B_{4R}|}{R^4} = 0. \end{aligned}$$

Therefore,

$$(3.30) \qquad \sum_{i=1}^{m} \int_{\mathbb{M}} \left(Ric_g(\nabla_g u_i, \nabla_g u_i) + |\mathcal{H}_{u_i}|^2 - |\nabla_g|\nabla_g u_i||^2 \right) dV_g$$

$$(3.31) \qquad + \sum_{i \neq j} \int_{\mathbb{M}} \left(|\partial_j H_i(u)||\nabla_g u_i||\nabla_g u_j| - \partial_j H_i(u)\nabla_g u_i \cdot \nabla_g u_j \right) dV_g \leq 0.$$

This implies that

$$(3.32) \qquad \sum_{i=1}^{m} \int_{\mathbb{M}} \left(Ric_g(\nabla_g u_i, \nabla_g u_i) + |\mathcal{H}_{u_i}|^2 - |\nabla_g|\nabla_g u_i||^2 \right) dV_g \leq 0.$$

The rest of the argument is very similar to the ones provided in the proof of Theorem 3.1.

$\square$

The ideas and methods applied in the above proof are strongly motivated by the ones provided in [14], when $\mathbb{M} = \mathbb{R}^n$, and in [16, 17] when the domain is a Riemannian manifold. Note that similar idea are used in [22] for the case of system of equations on $\mathbb{R}^n$. Lastly, in two dimensions, we have the following rigidity result for level sets of solutions. Note that as it is shown in [11], for the case of scalar equation, the following flatness result does not hold for the Allen-Cahn equation in $\mathbb{R}^n$ with $n \geq 9$ endowed with its standard flat metric. We assume that Ric_g is identically zero that is equivalent to the Gaussian curvature to be zero in two dimensions. We refer interested readers to [16, 17, 19, 35] for similar flatness results.

Theorem 3.4. *Suppose that $u = (u_i)_{i=1}^m$ is a stable solution of (1.1) and each $|\nabla_g u_i| \in L^\infty(\mathbb{M})$ where $\mathbb{M}$ is a Riemannian manifold with dim $\mathbb{M} = 2$. Assume that Ric_g is identically zero. Then, every connected component of any level set of each u_i, on which $|\nabla_g u_i|$ does not vanish, must be a geodesic.*

Proof. Note that assumptions imply that $\mathbb{M}$ is parabolic and it has nonnegative Gaussian curvature. Therefore, for each $p \in \mathbb{M}$ there exist a set M_p and a function f_ϵ such that

$$(3.33) \qquad \int_{\mathbb{M}} |\nabla_g f_\epsilon|^2 dV_g \leq \epsilon.$$

Set $\eta_i = f_\epsilon$ in the Poincaré inequality (2.8) to get

$$(3.34) \qquad \sum_{i=1}^{m} \int_{M_p} \left(Ric_g(\nabla_g u_i, \nabla_g u_i) + |\mathcal{H}_{u_i}|^2 - |\nabla_g|\nabla_g u_i||^2 \right) dV_g$$

$$+ \sum_{i \neq j} \int_{M_p} \left(|\partial_j H_i(u)||\nabla_g u_i||\nabla_g u_j| - \partial_j H_i(u)\nabla_g u_i \cdot \nabla_g u_j \right) dV_g$$

$$\leq \quad \epsilon \max_{i=1}^{m} \{ ||\nabla_g u_i||_{L^\infty(\mathbb{M})}^2 \}.$$

When ϵ approaches zero, we have

$$(3.35) \qquad \int_{M_p} Ric_g(\nabla_g u_i, \nabla_g u_i) dV_g = \int_{M_p} \left[|\mathcal{H}_{u_i}|^2 - |\nabla_g|\nabla_g u_i||^2 \right] dV_g = 0.$$

Therefore,

$$(3.36) \qquad |\nabla_g|\nabla_g u_i||^2(p) = |\mathcal{H}_{u_i}|^2(p),$$

for every $p \in M_p \cap \{\nabla_g u_i \neq 0\}$. Since the equality in (1.7) holds, there exists $\kappa_k : \mathbb{M} \to \mathbb{R}$ for each $k = 1, \cdots, n$ such that

$$(3.37) \qquad \nabla_g (\nabla_g u_i)_k(p) = \kappa_k(p)\nabla_g u_i(p).$$

For each $i = 1, \cdots, m$, consider a connected component Γ of $\{u_i \equiv C\} \cap \{\nabla_g u_i \neq 0\}$ that is a smooth curve. Let $\gamma_i : \mathbb{R} \to \mathbb{M}$ with

$$(3.38) \qquad |\dot{\gamma}_i|^2 = 1.$$

It is sufficient to show that

$$(3.39) \qquad \ddot{\gamma}_i^{\,k}(t) + \Gamma_{\lambda\mu}^k(p)\dot{\gamma}_i^{\,\lambda}(t)\dot{\gamma}_i^{\,\mu}(t) = 0 \;\; \text{in} \;\; \mathbb{R}.$$

We show that (3.39) holds for an arbitrary value $\bar{t} \in \mathbb{R}$ and $\bar{p} = \gamma_i(\bar{t})$. Differentiating (3.38) with respect to t we get

$$(3.40) \qquad 0 = \partial_k g_{\lambda\mu}(\gamma_i(t))\dot{\gamma}_i^{\,k}(t)\dot{\gamma}_i^{\,\lambda}(t)\dot{\gamma}_i^{\,\mu}(t) + 2g_{\lambda\mu}(\gamma_i(t))\dot{\gamma}_i^{\,k}(t)\ddot{\gamma}_i^{\,\lambda}(t).$$

We now use normal coordinates at some fixed point $\bar{p} \in \mathcal{M}$. Suppose that

$$(3.41) \qquad g_{\lambda\mu}(\bar{p}) = \delta_{\lambda\mu}, \;\; \partial_k g_{\lambda\mu}(\bar{p}) = 0 \;\; \text{and} \;\; \Gamma_{\lambda\mu}^k(\bar{p}) = 0.$$

Therefore,

$$(3.42) \qquad \dot{\gamma}_i(\bar{t})\ddot{\gamma}_i(\bar{t}) = 0.$$

Note that for any point on Γ we have $u_i(\gamma(t)) \equiv C$. From this we get

$$(3.43) \qquad 0 = \partial_\lambda u_i(\gamma(t))\dot{\gamma}_i^{\,\lambda}(t),$$

and

$$(3.44) \qquad 0 = \partial_{\lambda\mu} u_i(\gamma(t))\dot{\gamma}_i^{\,\lambda}(t)\dot{\gamma}_i^{\,\mu}(t) + \partial_\lambda u_i(\gamma(t))\ddot{\gamma}_i^{\,\lambda}(t).$$

Combining (3.44) and (3.37) for $p = \bar{p}$ and $k = \mu$, we end up with

$$(3.45) \qquad 0 = [\kappa_\mu(\bar{p})\dot{\gamma}_i^{\,\mu}(\bar{t})][\partial_\lambda u_i(\bar{p})\dot{\gamma}_i^{\,\lambda}(\bar{t})] + \partial_\lambda u_i(\bar{p})\ddot{\gamma}_i^{\,\lambda}(\bar{t}).$$

Substituting (3.43) in (3.45), we get

$$(3.46) \qquad 0 = \partial_\lambda u_i(\bar{p})\ddot{\gamma}_i^{\,\lambda}(\bar{t}).$$

From (3.46) and (3.42) we conclude that $\ddot{\gamma}_i^{\,\lambda}(\bar{t}) = 0$, since $\ddot{\gamma}_i^{\,\lambda}(\bar{t})$ is orthogonal to $\dot{\gamma}_i(\bar{t})$, that is tangent to $\{u_i \equiv C\}$ at $\bar{p}$, and to $\partial_\lambda u_i(\bar{p})$ that is orthogonal to $\{u_i \equiv C\}$ at $\bar{p}$. This and (3.41) imply that (3.39) holds at $\bar{t}$. This completes the proof.

$$\square$$

We end this section with the fact that for the most of our main results in this section, we assumed that the Ricci curvature is nonnegative. We would like to refer interested readers to [6], where hyperbolic spaces are discussed, and references therein for rigidity results when the curvature is negative.

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